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NON- ARCHIMEDEAN MENGER PM SPACE AND FIXED POINT

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ABSTRACT

By the help of probability distribution function function on set of positive real numbers Bharucha Reid [1] proved that a contraction mapping on a complete Menger Space has a unique fixed point under certain conditions By using the definition of weakly commuting pair of mappings with respect to other mapping, I proved the common fixed point theorem which is generalization of result of Sachdeva [11] in a metric space to more general setting of a non-Archimedean Menger space.

I. INTRODUCTION

The study of fixed points of sequence of mappings is initiated by Bonsall [2]. Let *F* be a family of self-mappings on a set *X*. An element $x \in X$ is called a fixed point of *F* if $fx = x$, for every $f \in F$. It is obvious that if a map $f \in F$ commutes with any $s \in F$ and has unique fixed point, then so is true for *F*. After studying the idea of Bonsall [2], Chatterjee [3] has also obtained some results on sequence of contraction mappings. Most of the papers dealing with fixed points for sequence of maps fall into one of three categories. The first category assumes that each pair f_i, f_j satisfies the same contractive condition and concludes that $\{f_n\}$ has a common fixed point. The second category assume that each f_n satisfies the same contractive condition and $\{f_n\}$ converges point wise to a limit function *f*. The conclusion is that *f* has a fixed-point*p*, which is the limit of each of the fixed points p_n of f_n . The third type assume that each f_n has a fixed point p_n, and that $\{f_n\}$ converges uniformly to a function which satisfies a particular contractive condition, with *p* is a fixed point of *f*. The conclusion is that $f_n \to p$.

Rhoades [10] extended a fourth class of theorems for a sequence of maps. In this the function f_i, f_j satisfy pair wise contractionprinciple, but with different contractive constants. The conclusion is that the sequence has a common unique fixed point.
The first result for a contractive self-mapping on a Menger PM space was obtained by Sehgal and Bharucha Reid [12]. Let (*X*, *F*) be PM space and $f : X \to X$ be a mapping. Then *f* is said to contraction if $\exists\ k \in (0\ 1)$ s.t. $\forall\ p,q \in X$, $F_{f(p)f(q)}(kx) \geq F_{pq}(x)$, $x > 0$.

II. PROBABILISTIC METRIC SPACES

DEFINITION [6]1.1. A mapping $f : R \to R^+$ is called a distribution function if it is non decreasing, left continuous and inf *f(x) = 0*, sup *f(x) = 1.*
We shall denote by *L* the set of all distribution functions. The specific distribution function $H \in L$ is defined by

$$H(x) = \left.\begin{aligned} &0, \quad x \leq 0 \\ &1, \quad x > 0 \end{aligned}\right\}$$

DEFINITION [6] 1.2 A probabilistic metric space (PM space) is an ordered pair (*X*,*F*), *X* is a nonempty set and $F : X \times X \to L$ is mapping such that, by denoting $F(p,q)$ by $F_{p,q}$ for all *p*, *q* in *X*, we have

(I) $F_{p,q}(x)=1\ \forall\ x>0$ iff $p=q$

(II) $F_{p,q}(0)=0$

(III) $F_{p,q}=F_{q,p}$

(IV) $F_{p,q}(x)=1$, $F_{q,r}(y)=1\Rightarrow F_{p,r}(x+y)=1.$

We note that $F_{p,q}(x)$ is value of the distribution function $F_{p,q}=F(p,q)\in L$ at $x\in R$.

DEFINITION [6] 1.3. A mapping $t:[0,1]\times[0,1]\rightarrow[0,1]$ is called t-norm if it is non-decreasing (by non-decreasing, we mean $a\le c, b\le d\Rightarrow t(a,b)\le t(c,d)$), commutative, associative and $t(a,1)=a$ for all *a* in [0, 1], *t*(0,0) = 0.

DEFINITION [6] 1.4. A Menger PM space is a triple (*X*, *F*; *t*) where (*X*, *F*) is a PM space and *t* is t-norm such that,
$F_{p,r}(x+y)\ge t\left(F_{p,q}(x),F_{q,r}(y)\right)\ \ \forall\ \ x,y\ge 0$.

NOTE [1]If (*X*, *F*; *t*) is Menger Probabilistic metric space with $\sup t\left(x,x\right)=1,\ 0<x<1$, then (*X*, *F; t*) is a Hausdorff topological space in the topology *T* induced by the family of (ε,λ) neighborhoods $\{U_p(\varepsilon,\lambda): p\in X, \in>0, \lambda>0\}$ where $U_p(\varepsilon,\lambda)=\{x\in X: F_{x,p}(\varepsilon)>1-\lambda\}$).

DEFINITION[6] 1.5 Suppose Ω = {g:g: [0,1] → [0,∞) is continuous, strictly increasing such that g(1)= 0, g(0) <∞} is a set of functions. A probabilistic metric space is said to be of type C_g if $g\in\Omega$ such that,

$g\left(F_{p,q}(x)\right)\le g\left(F_{p,r}(x)\right)+g\left(F_{r,q}(x)\right)\ \forall\ p,q,r\in X$ and $x\ge 0.$

NOTE. Throughout this paper we consider (*X*,*F*,*t*) a complete non- Archimedean Menger probabilistic metric space of type C_g .

Through this mapping every metric space can be considered as probabilistic metric space. For topological details the measure of compactness, completion, product and quotient spaces, refer to [4] and [8].
In 1977 Fisher [5] proved an interesting result on common fixed point theorem for a pair of self mappings on a complete metric space satisfying a contractive inequality.

THEOREM 1.1 [5]. Let *S* and *T* be mappings of a complete metric space (*X*, *d*) into itself satisfying the inequality,
$[d(Sp,Tq)]^2\le\alpha d(p,Sp)d(q,Tq)+\beta d(p,Tq)d(q,Sp)\ \ \forall p,q\in X,$
where $0\le\alpha<1$ and $\beta>0$ .Then *S* and *T* have a common fixed point. Further, if $0\le\alpha$, $\beta<1$ then each of *S* and *T* has a unique fixed common point.
In 1984 Rao and Rao [9] extended the above result of Fisher [5] for three mappings in the same setting of complete metric space.

THEOREM 1.2 [9]. Let *S*,*T* and *P* be mappings from a complete metric space (*X*, *d*) to itself satisfying the inequality,
$[d(SQp,TQq)]^2\le\alpha[d(p,q)]^2+\beta d(p,SQp)d(q,TQq)+\gamma d(p,TQq)d(q,SQp)\ \ \forall p,q\in X$

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where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta < 1$ and $\alpha + \gamma < 1$ and $SQ = QS$ or $TQ = QT$.
Then S,T and Q have a unique common fixed point.

After this Chatterjee and Singh [3] extended the above results of Fisher [5] and Rao and Rao [9] for four mappings without changing the setting.

In 1987, the existence of common fixed point theorems using the concept of weakly commuting pair of mappings with respect to certain mapping was introduced by Pathak [7].

DEFINITION 1.6[7]. Let P, S and Tbe mappings from a complete metric space (X, d) to itself. Then $\{S, T\}$ is said to be weakly commuting pair of mappings with respect to mapping P if,

$$d(PSPx, TPx) \leq d(SPPx, TPx)$$

and

$$d(SPx, PTPx) \leq d(SPx, TPPx),$$

for all x in X.

Now we define weakly commuting pair of mappings with respect to certain mapping in non- Archimedean Menger space of type C_g.

DEFINITION 1.7 Let (X, F, t) be a non-Archimedean Menger space of type C_g. Suppose S, T, P are self mappings on X. Then the pair (S, T) is said to be weakly commuting pair with respect to P if $\forall\ t \in X$,

$$g(F_{PSPx,TPx}(t)) \leq g(F_{SPPx,TPx}(t))$$

and

$$g(F_{PTPx,SPx}(t)) \leq g(F_{TPPx,SPx}(t)).$$

The above theorem and results of Tripathi *et* al.[13] and [14] promoted us to further generalizing some of the results. Subsequently, using commuting- map concept, a variety of variations and generalizations of the above theorem were obtained by

III. MAIN RESULTS

THEOREM 2.1 Let (X, F, t) be a complete non- Archimedean Menger space. Suppose S, T, P are self mappings on X satisfying,

(1). $$[g(F_{SPx,TPy}(t))]^2 \geq \alpha_1[g(F_{x,y}(t))]^2 + \alpha_2 g(F_{x,SPx}(t))g(F_{y,TPy}(t)) + \alpha_3 g(F_{x,TPy}(t))g(F_{y,SPx}(t))$$

$$+\alpha_4 g(F_{x,SPx}(t))g(F_{x,TPy}(t)) + \alpha_5 g(F_{y,TPy}(t))g(F_{y,SPx}(t))$$

for $\alpha_i > 0, \quad i = 1, 2, \ldots 5$ such that

$$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + < 1.$$

(2). $SP = PS$ or $TP = PT$.
Then S, T, P have a unique common fixed point in X.

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PROOF. For $x_0 \in X$, we construct a sequence $\{x_n\}$ such that,

$x_{2n+1} = SPx_{2n}$ and $x_{2n} = TPx_{2n-1}$, $n = 1, 2...$

Then from (1),

$$[g(F_{x_{2n+1},x_{2n}}(t))]^2 = [g(F_{SPx_{2n},TPx_{2n-1}}(t))]^2$$

i.e. $[g(F_{x_{2n+1},x_{2n}}(t))]^2 \geq \alpha_1[g(F_{x_{2n},x_{2n-1}}(t))]^2 + \alpha_2 g(F_{x_{2n},x_{2n+1}}(t))g(F_{x_{2n-1},x_{2n}}(t)) + \alpha_3 g(F_{x_{2n},x_{2n}}(t))g(F_{x_{2n-1},x_{2n+1}}(t))$

$$+\alpha_4 g(F_{x_{2n},x_{2n+1}}(t))g(F_{x_{2n},x_{2n}}(t)) + \alpha_5 g(F_{x_{2n-1},x_{2n}}(t))g(F_{x_{2n-1},x_{2n+1}}(t))$$

i.e. $[g(F_{x_{2n+1},x_{2n}}(t))]^2 \geq \alpha_1[g(F_{x_{2n},x_{2n-1}}(t))]^2 + (\alpha_2)g(F_{x_{2n},x_{2n+1}}(t))g(F_{x_{2n-1},x_{2n}}(t))$

$$+(\alpha_5)g(F_{x_{2n},x_{2n-1}}(t))g(F_{x_{2n-1},x_{2n+1}}(t))$$

i.e. $[g(F_{x_{2n+1},x_{2n}}(t))]^2 \geq \alpha_1[g(F_{x_{2n},x_{2n-1}}(t))]^2 + \alpha_2 g(F_{x_{2n},x_{2n+1}}(t))g(F_{x_{2n-1},x_{2n}}(t))$

$$+(\alpha_5)g(F_{x_{2n},x_{2n-1}}(t))[g(F_{x_{2n},x_{2n+1}}(t)) + g(F_{x_{2n},x_{2n-1}}(t))]$$

or $[g(F_{x_{2n+1},x_{2n}}(t))]^2 \geq (\alpha_1 + \alpha_5)[g(F_{x_{2n},x_{2n-1}}(t))]^2 + (\alpha_5 + \alpha_2)g(F_{x_{2n},x_{2n+1}}(t))g(F_{x_{2n-1},x_{2n}}(t))$

or $[g(F_{x_{2n+1},x_{2n}}(t))]^2 \geq (\alpha_1 + \alpha_5)[g(F_{x_{2n},x_{2n-1}}(t))]^2 + (\alpha_5 + \alpha_2)\dfrac{[\{g(F_{x_{2n},x_{2n+1}}(t))\}^2 + \{g(F_{x_{2n-1},x_{2n}}(t))\}^2]}{2}$

i.e. $[g(F_{x_{2n+1},x_{2n}}(t))]^2 \geq \dfrac{\{(\alpha_1 + \alpha_5) + \frac{1}{2}(\alpha_5 + \alpha_2)\}}{1 - \frac{1}{2}(\alpha_5\alpha_2)} g(F_{x_{2n-1},x_{2n}}(t))$

i.e. $[g(F_{x_{2n+1},x_{2n}}(t))] \geq k(F_{x_{2n-1},x_{2n}}(t))$,

where $k^2 = \dfrac{(\alpha_1 + \alpha_5) + \frac{1}{2}(\alpha_5 + \alpha_2)}{1 - \frac{1}{2}(\alpha_5 + \alpha_2)} < 1,$

(because if $k^2 \geq 1$, then $\alpha_1 + \alpha_2 + 2\alpha_5 \geq 1,$ which contradicts our assumption).

Hence Inductively,

$$[g(F_{x_{2n+1},x_{2n}}(t))] \geq k^n(F_{x_n,x_{n+1}}(t)).$$

Now in limiting case as *n*, we have,

$$\{g(F_{x_{2n+1},x_{2n}}(t)\} \to 0.$$

i.e. $\lim_{n\to\infty} g(F_{x_n,x_{n+1}}(t)\} = 0.$

Next we show that $\{x_n\}$ is a Cauchy sequence.

If $\{x_n\}$ is not a Cauchy sequence then $\exists\ \varepsilon_0 > 0$, $t_0 > 0$ and sets of positive integers $\{m_i\}, \{n_i\}$ such that,

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$\lim_{i\to\infty} g(F_{x_{m_i},x_{n_i}}(t_0)) = g(1-\varepsilon_0)$,

$\lim_{i\to\infty} g(F_{x_{m_i-1},x_{n_i-1}}(t_0)) = g(1-\varepsilon_0)$,

$\lim_{i\to\infty} g(F_{x_{m_i+1},x_{n_i-1}}(t_0)) = g(1-\varepsilon_0)$

and $\lim_{i\to\infty} g(F_{x_{m_i},x_{n_i-1}}(t_0)) = g(1-\varepsilon_0)$.

Putting $x = x_{2m_i}$ and $y = x_{2n_i-1}$ in (1), we get,

$$[g(F_{SPx_{2m_i},TPx_{2n_i-1}}(t))]^2 \geq \alpha_1[g(F_{x_{2m_i},x_{2n_i-1}}(t))]^2 + \alpha_2 g(F_{x_{2m_i},x_{2m_i+1}}(t))g(F_{x_{2n_i-1},x_{2n_i}}(t)) + \alpha_3 g(F_{x_{2m_i},x_{2n_i}}(t))g(F_{x_{2n_i-1},x_{2m_i+1}}(t))$$
$$+\alpha_4 g(F_{x_{2m_i},x_{2m_i+1}}(t))g(F_{x_{2m_i},x_{2n_i}}(t)) + \alpha_5 g(F_{x_{2n_i-1},x_{2n_i}}(t))g(F_{x_{2n_i-1},x_{2m_i+1}}(t))$$

.

Taking limit as $n \to \infty$, we have,

$[g(1-\varepsilon_0)]^2 \leq (\alpha_1+\alpha_3)[g(1-\varepsilon_0)]^2$

i.e. $[g(1-\varepsilon_0)]^2 < [g(1-\varepsilon_0)]^2$

(because $(\alpha_1+\alpha_3) < 1$), which is not possible. Thus $\{x_n\}$ is a Cauchy sequence. Since $(X, F; t)$ be a complete non-Archimedean Menger space so

$\{x_n\} \to z \in X$ hence from (1), we get,

$[g(F_{SPz,x_{2n}}(t))]^2 = [g(F_{SPz,TPx_{2n-1}}(t))]^2$,

i.e. $$[g(F_{SPz,x_{2n}}(t))]^2 \geq \alpha_1[g(F_{z,x_{2n-1}}(t))]^2 + \alpha_2 g(F_{z,SPz}(t))g(F_{x_{2n-1},x_{2n}}(t)) + \alpha_3 g(F_{z,x_{2n}}(t))g(F_{x_{2n-1},SPz}(t))$$
$$+\alpha_4 g(F_{z,SPz}(t))g(F_{z,x_{2n}}(t)) + \alpha_5 g(F_{x_{2n-1},x_{2n}}(t))g(F_{x_{2n-1},SPz}(t))$$

Making $n \to \infty$, we get,

$[g(F_{SPz,z}(x))]^2 \leq 0$ i.e. $g(F_{SPz,z}(x)) = 0$ or $SPz = z$.

Similarly by considering $[g(F_{x_{2n+1},TPz}(t))]^2$, we conclude from (1) that

$TPz = z$, i.e. $TPz = z = SPz$ … (2.2).

Again if $SP = PS$, then

$[g(F_{Pz,z}(t))]^2 = [g(F_{PSPz,TPz}(t))]^2 = [g(F_{SPPz,TPz}(t))]^2$ … (2.3)

i.e.

$$[g(F_{Pz,z}(t))]^2 \geq \alpha_1[g(F_{Pz,z}(t))]^2 + \alpha_2 g(F_{Pz,Pz}(t))g(F_{z,z}(t)) + \alpha_3 g(F_{Pz,z}(t))g(F_{z,Pz}(t))$$
$$+\alpha_4 g(F_{Pz,Pz}(t))g(F_{Pz,z}(t)) + \alpha_5 g(F_{z,z}(t))g(F_{z,Pz}(t))$$

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i.e. $[g(F_{Pz,z}(t))]^2 \le (\alpha_1+\alpha_3)[g(F_{Pz,z}(t))]^2$,

i.e. $(1-(\alpha_1+\alpha_3+\alpha_7+\alpha_9+\alpha_{10}))[g(F_{Pz,z}(t))]^2 \le 0$, hence $g(F_{Pz,z}(t))=0$, thus $Pz = z$.

Hence by (2.2) $Tz = z = Sz$. Similarly if $PT = TP$, then also $Pz = z = Sz = Tz$.
Therefore z is a common fixed point of S, T, P.
For uniqueness suppose z and z' are common fixed points of S, T, P.
Then from (1), we have,

$$[g(F_{z,z'}(t))]^2 = [g(F_{SPz,TPz'}(t))]^2 \le \alpha_1[g(F_{z,z'}(t))]^2 + \alpha_2 g(F_{z,z}(t))g(F_{z',z'}(t)) + \alpha_3 g(F_{z,z'}(t))g(F_{z',z}(t))$$

$$+\alpha_4 g(F_{z,z}(t))g(F_{z,z}(t)) + \alpha_5 g(F_{z',z'}(t))g(F_{z',z'}(t))$$

i.e. $[g(F_{z,z'}(t))]^2 \ge (\alpha_1+\alpha_3)[g(F_{z,z'}(t))]^2$

i.e. $(1-(\alpha_1+\alpha_3))[g(F_{z,z'}(t))]^2 \ge 0$, therefore $g(F_{z,z'}(t))=0$, hence $z = z'$.

This proves the uniqueness.

By using the definition of weakly commuting pair of mappings with respect to other mapping, we have proved the following theorem which is generalization of result of Sachdeva [169] in a metric space to more general setting of a non-Archimedean menger space of type C_g.

THEOREM 2.2. Let (X, F; t) be a complete non- Archimedean Menger space. Suppose S, T, P are self mappings on X satisfying the following conditions.

(1). $$[g(F_{SPx,TPy}(t))]^2 \ge \alpha_1[g(F_{x,y}(t))]^2 + \alpha_2 g(F_{x,SPx}(t))g(F_{y,TPy}(t)) + \alpha_3 g(F_{x,TPy}(t))g(F_{y,SPx}(t))$$

$$+\alpha_4 g(F_{x,SPx}(t))g(F_{x,TPy}(t)) + \alpha_5 g(F_{y,TPy}(t))g(F_{y,SPx}(t))$$

for all $\alpha_i > 0,\ i = 1,2,...5$ such that

$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+<1$,

(2). The pair (S, T) is weakly commuting with respect to P.
Then S, T, P have a unique common fixed point in X.

PROOF. By proceeding as in the proof of above Theorem 2.1, we conclude that z is a common fixed point of SP and TP , i.e. $SPz = TPz = z$.

Now, using the condition (1) and definition of weakly ommuting pair (S,T) with respect to P, we have,

$[g(F_{Pz,z}(t))]^2 = [g(F_{PSPz,TPz}(t))]^2 \le [g(F_{SPPz,TPz}(t))]^2.$

Again as in the proof of above theorem along with (2.3), z is unique common fixed point of S, T and P.

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